

# Approximate symmetries of geodesic equations on 2-spheres

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**ABSTRACT:** Approximate symmetries of geodesic equations on 2-spheres are studied. These are the symmetries of the perturbed geodesic equations which represent approximate path of a particle rather than exact path. After giving the exact symmetries of the geodesic equations, two different approaches to study the approximate symmetries of the approximate geodesic equations show that no non-trivial approximate symmetry for these spaces exists.

# 1. Introduction

For nonlinear problems, analytic solutions are rare and hard to obtain. Lie group theory provide a systematic and unified approach in search of analytical solutions [1, 2, 3]. Instead of solving the nonlinear equations directly, a system of over determined linear equations is studied using the theory. Once the so-called symmetries of the equations are calculated, similarity solutions can be produced. By defining canonical coordinates, it is also possible to transform the equations into a much simpler form. Another approach allows to find approximate symmetries of approximate differential equations. Generally, the perturbed term in a differential equation corresponds to some small error or correction. Therefore the resulting equations are defined approximately depending on a small parameter and such equations occur frequently in applications. The theory of approximate groups provides a regular method of calculating the perturbation directly, without using the complicated group transformations. The calculation is based on approximate symmetry equations. Recently, approximate symmetries have been used to find the approximate solutions of some partial differential equations [4, 5] as well.

Let us consider an approximate (perturbed) equation in variables  $t$  and  $y$ , with a small parameter  $\epsilon$ , i.e.

$$E(t, y, \epsilon) \approx 0. \quad (1.1)$$

We write it as

$$E(t, y, \epsilon) \equiv E_0(t, y) + \epsilon E_1(t, y) \approx 0, \quad (1.2)$$

where  $E_0$  and  $E_1$  are respectively the exact and approximate parts of the approximate differential equation (1.1). For (1.2) the infinitesimal generator can be written as [1, 2]

$$\mathbf{X} = \mathbf{X}_0 + \epsilon \mathbf{X}_1, \quad (1.3)$$

where

$$\begin{aligned} \mathbf{X}_0 &= \xi_0 \frac{\partial}{\partial t} + \eta_0 \frac{\partial}{\partial y}, \\ \mathbf{X}_1 &= \xi_1 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial y}. \end{aligned}$$

Here  $\xi_0$ ,  $\xi_1$ ,  $\eta_0$  and  $\eta_1$  are all functions of  $t$  and  $y$ .  $X_0$  is the generator of the exact differential equation  $E_0$ , and  $X_1$  is the generator of the approximate differential equation  $E_1$ . Hence we can write

$$\mathbf{X} = (\xi_0 + \epsilon \xi_1) \frac{\partial}{\partial t} + (\eta_0 + \epsilon \eta_1) \frac{\partial}{\partial y}. \quad (1.4)$$

More generally

$$\mathbf{X} = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}, \quad (1.5)$$

where

$$\begin{aligned} \xi &= \xi_0 + \epsilon \xi_1, \\ \eta &= \eta_0 + \epsilon \eta_1. \end{aligned}$$

A second prolongation [1, 2] is given by

$$\mathbf{X} = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y} + \eta^{(1)} \frac{\partial}{\partial y'} + \eta^{(2)} \frac{\partial}{\partial y''}. \quad (1.6)$$

Equation (1.2) is approximately invariant under the approximate group of transformations with the generator given in (1.3) if and only if  $\mathbf{X}E|_{E \approx 0} = o(\epsilon)$  [2], or

$$[\mathbf{X}_0 E_0 + \epsilon(\mathbf{X}_0 E_1 + \mathbf{X}_1 E_0)]|_{E \approx 0} = o(\epsilon). \quad (1.7)$$

This is the *determining equation* and  $X$  is called an *infinitesimal approximate symmetry* or an *approximate symmetry* or an *approximate operator* [2], admitted by (1.2).

If  $X_0$  is a generator of the unperturbed differential equation

$$E_0 = 0, \quad (1.8)$$

obtained by solving the determining equation

$$\mathbf{X}_0 E_0|_{E \approx 0} = 0, \quad (1.9)$$

then we define the auxiliary function  $H$  by

$$H = \frac{1}{\epsilon} \mathbf{X}_0 (E_0 + \epsilon E_1)|_{E_0 + \epsilon E_1 = 0}, \quad (1.10)$$

and find an *approximate symmetry*, (1.3), of the perturbed differential equation (1.2) by solving for  $X_1$  [2] in

$$\mathbf{X}_1 E_0|_{E \approx 0} + H = 0. \quad (1.11)$$

Note that (1.11), unlike the determining equation (1.9) for exact symmetries, is non-homogeneous.

The geodesic equations on a manifold can be regarded as a system of second order ordinary differential equations. The curvature of surfaces can be positive, zero

or negative. Symmetries of geodesic equations for these three types of manifolds have been studied [6]. Interesting connection between these symmetries and the isometries [7] of these spaces have also been found. In this paper we explore approximate symmetries of geodesic equations of 2-spheres. Perturbations in geodesic equations correspond to approximate path of the particle as opposed to the exact one. It may be noted that we study approximate geodesics on a 2-manifold rather than exact geodesics on a perturbed manifold.

## 2. Approximate symmetries of geodesic equations on a 2-sphere

A geodesic is a locally length-minimizing curve and give the shortest distance between two points. Equivalently, it is a path that a particle which is not accelerating would follow. The geodesic equations on a manifold can be regarded as a system of second order ordinary differential equations. In coordinates  $x^a$  these can be expressed as [8, 9]

$$\ddot{x}^a + \Gamma_{bc}^a \dot{x}^b \dot{x}^c = 0, \quad (2.1)$$

where dot represents derivative with respect to the arc length parameter  $s$  and  $\Gamma_{bc}^a$  denote the Christoffel symbols, which for a metric  $g_{ab}$ , are defined by

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (g_{bc,d} + g_{bd,c} - g_{cd,b}). \quad (2.2)$$

Here  $g^{ab}$  denotes the inverse of the metric and the range of the indices gives the dimension of the space.

Exact symmetries of the unit sphere have been given in Ref. [6]. We first outline the derivation of these symmetries and then use them for constructing the approximate symmetries. The metric for the sphere can be written as

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (2.3)$$

The geodesic equations (2.1) for this metric are given by

$$\begin{aligned} E_1 : \ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 &= 0, \\ E_2 : \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} &= 0. \end{aligned} \quad (2.4)$$

These are second order ordinary differential equations. We apply the second prolongation [6],

$$\mathbf{X} = \xi \frac{\partial}{\partial s} + \eta^1 \frac{\partial}{\partial \theta} + \eta^2 \frac{\partial}{\partial \phi} + \eta_{,s}^1 \frac{\partial}{\partial \dot{\theta}} + \eta_{,s}^2 \frac{\partial}{\partial \dot{\phi}} + \eta_{,ss}^1 \frac{\partial}{\partial \ddot{\theta}} + \eta_{,ss}^2 \frac{\partial}{\partial \ddot{\phi}}, \quad (2.5)$$

of the symmetry generator given by (1.3), to both the geodesic equations. Here  $\xi$ ,  $\eta^1$  and  $\eta^2$  are all functions of  $s$ ,  $\theta$  and  $\phi$ ;  $\eta_{,s}^1$  and  $\eta_{,s}^2$  are functions of  $s$ ,  $\theta$ ,  $\phi$ ,  $\dot{\theta}$  and  $\dot{\phi}$ ; and  $\eta_{,ss}^1$  and  $\eta_{,ss}^2$  are all functions of  $s$ ,  $\theta$ ,  $\phi$ ,  $\dot{\theta}$ ,  $\dot{\phi}$ ,  $\ddot{\theta}$  and  $\ddot{\phi}$ . Then

$$\mathbf{X}E_1 \big|_{E_1=0=E_2} = 0,$$

$$\mathbf{X}E_2 \big|_{E_1=0=E_2} = 0,$$

respectively yield [6]

$$\left[ \eta_{,ss}^1 - 2 \sin \theta \cos \theta \dot{\phi} \eta_{,s}^2 - \eta^1 (\cos^2 \theta - \sin^2 \theta) \dot{\phi}^2 \right] \big|_{E_1=0=E_2} = 0, \quad (2.6)$$

$$\left[ \eta_{,ss}^2 + 2 \cot \theta \left( \dot{\phi} \eta_{,s}^1 + \dot{\theta} \eta_{,s}^2 \right) - 2 \dot{\theta} \dot{\phi} \csc^2 \theta \eta^1 \right] \big|_{E_1=0=E_2} = 0. \quad (2.7)$$

If we write

$$D = \frac{\partial}{\partial s} + \dot{\theta} \frac{\partial}{\partial \theta} + \dot{\phi} \frac{\partial}{\partial \phi} + \ddot{\theta} \frac{\partial}{\partial \dot{\theta}} + \ddot{\phi} \frac{\partial}{\partial \dot{\phi}}, \quad (2.8)$$

then

$$\eta_{,s}^1 = D\eta^1 - \dot{\theta} D\xi = \eta_s^1 + \dot{\theta} (\eta_\theta^1 - \xi_s) - \dot{\theta}^2 \xi_\theta + \dot{\phi} \eta_\phi^1 - \dot{\theta} \dot{\phi} \xi_\phi, \quad (2.9)$$

$$\eta_{,s}^2 = D\eta^2 - \dot{\phi} D\xi = \eta_s^2 + \dot{\phi} (\eta_\phi^2 - \xi_s) + \dot{\theta} \eta_\theta^2 - \dot{\phi}^2 \xi_\phi - \dot{\theta} \dot{\phi} \xi_\theta, \quad (2.10)$$

$$\begin{aligned} \eta_{,ss}^1 &= D\eta_{,s}^1 - \ddot{\theta} D\xi \\ &= \eta_{ss}^1 + \dot{\theta} (2\eta_{s\theta}^1 - \xi_{ss}) + \dot{\theta}^2 (\eta_{\theta\theta}^1 - 2\xi_{s\theta}) - \dot{\theta}^3 \xi_{\theta\theta} + 2\dot{\phi} \eta_{s\phi}^1 \\ &\quad + \dot{\phi}^2 \eta_{\phi\phi}^1 + 2\dot{\theta} \dot{\phi} (\eta_{\theta\phi}^1 - \xi_{s\phi}) - 2\dot{\theta}^2 \dot{\phi} \xi_{\theta\phi} - \dot{\theta} \dot{\phi}^2 \xi_{\phi\phi} \\ &\quad + \ddot{\theta} (\eta_\theta^1 - 2\xi_s - 3\dot{\theta} \xi_\theta) - 2\ddot{\theta} \dot{\phi} \xi_\phi + \ddot{\phi} \eta_\phi^1 - \dot{\theta} \ddot{\phi} \xi_\phi, \end{aligned} \quad (2.11)$$

$$\begin{aligned} \eta_{,ss}^2 &= D\eta_{,s}^2 - \ddot{\phi} D\xi \\ &= \eta_{ss}^2 + 2\dot{\theta} \eta_{s\theta}^2 + \dot{\theta}^2 \eta_{\theta\theta}^2 + \dot{\phi} (2\eta_{s\phi}^2 - \xi_{ss}) + \dot{\phi}^2 (\eta_{\phi\phi}^2 - 2\xi_{s\phi}) \\ &\quad - \dot{\phi}^3 \xi_{\phi\phi} + 2\dot{\theta} \dot{\phi} (\eta_{\theta\phi}^2 - \xi_{s\theta}) - \dot{\theta}^2 \dot{\phi} \xi_{\theta\theta} - 2\dot{\theta} \dot{\phi}^2 \xi_{\theta\phi} + \ddot{\theta} \eta_\theta^2 \\ &\quad - \ddot{\theta} \dot{\phi} \xi_\theta + \ddot{\phi} (\eta_\phi^2 - 2\xi_s - 3\dot{\phi} \xi_\phi) - 2\ddot{\phi} \dot{\theta} \xi_\theta. \end{aligned} \quad (2.12)$$

Inserting (2.9) - (2.12) into (2.6) and (2.7) yields

$$\begin{aligned} &[\eta_{ss}^1 + \dot{\theta} (2\eta_{s\theta}^1 - \xi_{ss}) + \dot{\theta}^2 (\eta_{\theta\theta}^1 - 2\xi_{s\theta}) - \dot{\theta}^3 \xi_{\theta\theta} + 2\dot{\phi} \eta_{s\phi}^1 + \dot{\phi}^2 \eta_{\phi\phi}^1 \\ &\quad + 2\dot{\theta} \dot{\phi} (\eta_{\theta\phi}^1 - \xi_{s\phi}) - 2\dot{\theta}^2 \dot{\phi} \xi_{\theta\phi} - \dot{\theta} \dot{\phi}^2 \xi_{\phi\phi} + \ddot{\theta} (\eta_\theta^1 - 2\xi_s - 3\dot{\theta} \xi_\theta) \\ &\quad - 2\ddot{\theta} \dot{\phi} \xi_\phi + \ddot{\phi} \eta_\phi^1 - \dot{\theta} \ddot{\phi} \xi_\phi] - \sin 2\theta \dot{\phi} [\eta_s^2 + \dot{\phi} (\eta_\phi^2 - \xi_s) + \dot{\theta} \eta_\theta^2 - \dot{\phi}^2 \xi_\phi \\ &\quad - \dot{\theta} \dot{\phi} \xi_\theta] - \eta^1 \cos 2\theta \dot{\phi}^2 \big|_{E_1=0=E_2} = 0, \end{aligned} \quad (2.13)$$

$$\begin{aligned}
& [\eta_{ss}^2 + 2\dot{\theta}\eta_{s\theta}^2 + \dot{\theta}^2\eta_{\theta\theta}^2 + \dot{\phi}(2\eta_{s\phi}^2 - \xi_{ss}) + \dot{\phi}^2(\eta_{\phi\phi}^2 - 2\xi_{s\phi}) - \dot{\phi}^3\xi_{\phi\phi} \\
& + 2\dot{\theta}\dot{\phi}(\eta_{\theta\phi}^2 - \xi_{s\theta}) - \dot{\theta}^2\dot{\phi}\xi_{\theta\theta} - 2\dot{\theta}\dot{\phi}^2\xi_{\theta\phi} + \ddot{\theta}\eta_{\theta}^2 - \ddot{\theta}\dot{\phi}\xi_{\theta} + \ddot{\phi}(\eta_{\phi}^2 - 2\xi_s - 3\dot{\phi}\xi_{\phi}) \\
& - 2\dot{\theta}\ddot{\phi}\xi_{\theta}] + 2\cot\theta(\dot{\phi}\{\eta_s^1 + \dot{\theta}(\eta_{\theta}^1 - \xi_s) - \dot{\theta}^2\xi_{\theta} + \dot{\phi}\eta_{\phi}^1 - \dot{\theta}\dot{\phi}\xi_{\phi}\} \\
& + \dot{\theta}\{\eta_s^2 + \dot{\phi}(\eta_{\phi}^2 - \xi_s) + \dot{\theta}\eta_{\theta}^2 - \dot{\phi}^2\xi_{\phi} - \dot{\theta}\dot{\phi}\xi_{\theta}\}) - 2\dot{\theta}\dot{\phi}\csc^2\theta\eta^1|_{E_1=0=E_2} = 0. \quad (2.14)
\end{aligned}$$

Inserting the values of  $\ddot{\theta}$  and  $\ddot{\phi}$  from the geodesic equations (2.4), and then comparing the coefficients of the powers of  $\dot{\theta}$  and  $\dot{\phi}$ , we obtain the following system of partial differential equations.

$$(\dot{\theta}\dot{\phi})^0 : \eta_{ss}^1 = 0, \quad (2.15)$$

$$\eta_{ss}^2 = 0, \quad (2.16)$$

$$\dot{\theta} : 2\eta_{s\theta}^1 - \xi_{ss} = 0, \quad (2.17)$$

$$\eta_{s\theta}^2 + \cot\theta\eta_s^2 = 0, \quad (2.18)$$

$$\dot{\phi} : 2\eta_{s\phi}^1 - \sin 2\theta\eta_s^2 = 0, \quad (2.19)$$

$$2\eta_{s\phi}^2 - \xi_{ss} + 2\cot\theta\eta_s^1 = 0, \quad (2.20)$$

$$\dot{\theta}^2 : \eta_{\theta\theta}^1 - 2\xi_{s\theta} = 0, \quad (2.21)$$

$$\eta_{\theta\theta}^2 + 2\cot\theta\eta_{\theta}^2 = 0, \quad (2.22)$$

$$\dot{\phi}^2 : \eta_{\phi\phi}^1 + \sin\theta\cos\theta\eta_{\theta}^1 - \eta^1\cos 2\theta - \sin 2\theta\eta_{\phi}^2 = 0, \quad (2.23)$$

$$\eta_{\phi\phi}^2 - 2\xi_{s\phi} + \sin\theta\cos\theta\eta_{\theta}^2 + 2\cot\theta\eta_{\phi}^1 = 0, \quad (2.24)$$

$$\dot{\theta}^3 : \xi_{\theta\theta} = 0, \quad (2.25)$$

$$\dot{\phi}^3 : -\xi_{\phi\phi} - \sin\theta\cos\theta\xi_{\theta} = 0, \quad (2.26)$$

$$\dot{\theta}\dot{\phi} : 2(\eta_{\theta\phi}^1 - \xi_{s\phi}) - \sin 2\theta\eta_{\theta}^2 - 2\cot\theta\eta_{\phi}^1 = 0, \quad (2.27)$$

$$(\eta_{\theta\phi}^2 - \xi_{s\theta}) + \cot\theta\eta_{\theta}^1 - \csc^2\theta\eta^1 = 0, \quad (2.28)$$

$$\dot{\theta}\dot{\phi}^2 : -\xi_{\phi\phi} - \sin\theta\cos\theta\xi_{\theta} = 0, \quad (2.29)$$

$$-2\xi_{\theta\phi} + 2\cot\theta\xi_{\phi} = 0. \quad (2.30)$$

Equations (2.25), (2.26) and (2.29) yield

$$\xi = b_1(s). \quad (2.31)$$

Substituting the value of  $\xi$  into (2.21) and solving with (2.15), we get

$$\eta^1 = [f_1(\phi)s + f_2(\phi)]\theta + f_3(\phi)s + f_4(\phi). \quad (2.32)$$

From (2.16), (2.18) and (2.22) we obtain

$$\eta^2 = -\cot\theta f_6(\phi) + f_7(\phi). \quad (2.33)$$

Substituting the values of  $\eta^1$  and  $\eta^2$  into (2.19) we get

$$\eta^1 = [As + f_2(\phi)]\theta + Bs + f_4(\phi). \quad (2.34)$$

Then (2.17) yields

$$\xi = As^2 + c_1s + c_0. \quad (2.35)$$

Now (2.23) and (2.27) give

$$\xi = c_1s + c_0, \quad (2.36)$$

$$\eta^1 = c_3 \cos \phi + c_4 \sin \phi, \quad (2.37)$$

$$\eta^2 = \cot \theta (c_4 \cos \phi - c_3 \sin \phi) + c_2. \quad (2.38)$$

Therefore, the exact symmetries come out to be

$$\begin{aligned} \mathbf{X}_0 &= \frac{\partial}{\partial s}, \mathbf{X}_1 = s \frac{\partial}{\partial s}, \mathbf{X}_2 = \frac{\partial}{\partial \phi}, \\ \mathbf{X}_3 &= \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \\ \mathbf{X}_4 &= \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}. \end{aligned} \quad (2.39)$$

In the next sections we discuss two different approaches for finding approximate symmetries of this 2-manifold.

## 2.1 Approximate symmetries: First approach

Here we will convert the geodesic equations (2.4) of the sphere into perturbed equations by adding a general function and then try to find the approximate symmetries of these equations.

The second prolongations of the five exact symmetries (2.39) of the geodesic equations of the sphere are given below.

$$(i) \quad \mathbf{X}^0 = \frac{\partial}{\partial s}$$

Here  $\xi = 1$ ,  $\eta^1 = 0 = \eta^2$ , therefore, equations (2.9) -(2.12) give  $\eta_{,s}^1 = 0 = \eta_{,s}^2$ ,  $\eta_{,ss}^1 = 0 = \eta_{,ss}^2$  so that the second prolongation becomes

$$\mathbf{X}^0 = \frac{\partial}{\partial s}. \quad (2.40)$$

$$(ii) \quad \mathbf{X}^1 = s \frac{\partial}{\partial s}$$

Here  $\xi = s$ ,  $\eta^1 = 0 = \eta^2$ , therefore, equations (2.9) -(2.12) give  $\eta_{,s}^1 = -\dot{\theta}$ ,  $\eta_{,s}^2 = -\dot{\phi}$ ,  $\eta_{,ss}^1 = -2\ddot{\theta}$ ,  $\eta_{,ss}^2 = -2\ddot{\phi}$ , so that the second prolongation becomes

$$\mathbf{X}^1 = s \frac{\partial}{\partial s} - \dot{\theta} \frac{\partial}{\partial \theta} - \dot{\phi} \frac{\partial}{\partial \phi} - 2\ddot{\theta} \frac{\partial}{\partial \ddot{\theta}} - 2\ddot{\phi} \frac{\partial}{\partial \ddot{\phi}}. \quad (2.41)$$

$$(iii) \quad \mathbf{X}^2 = \frac{\partial}{\partial \phi}$$

Here  $\xi = 0$ ,  $\eta^1 = 0$ ,  $\eta^2 = 1$ , therefore, equations (2.9) -(2.12) give  $\eta_{,s}^1 = 0 = \eta_{,s}^2$ ,  $\eta_{,ss}^1 = 0 = \eta_{,ss}^2$  so that second prolongation becomes

$$\mathbf{X}^2 = \frac{\partial}{\partial \phi}. \quad (2.42)$$

$$(iv) \quad \mathbf{X}^3 = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} 0.$$

Here  $\xi = 0$ ,  $\eta^1 = \cos \phi$ ,  $\eta^2 = -\cot \theta \sin \phi$ , therefore, equations (2.9) -(2.12) give

$$\begin{aligned} \eta_{,s}^1 &= -\dot{\phi} \sin \phi, \\ \eta_{,s}^2 &= \dot{\theta} (\sin \phi \csc^2 \theta) + \dot{\phi} (-\cot \theta \cos \phi), \\ \eta_{,ss}^1 &= -\dot{\phi}^2 \cos \phi - \ddot{\phi} \sin \phi, \\ \eta_{,ss}^2 &= \dot{\theta}^2 \sin \phi (-2 \csc^2 \theta \cot \theta) + \dot{\phi}^2 \cot \theta \sin \phi + 2\dot{\theta}\dot{\phi} (\cos \phi \csc^2 \theta) \\ &\quad + \ddot{\theta} \csc^2 \theta \sin \phi + \ddot{\phi} (-\cot \theta \cos \phi), \end{aligned}$$

so that the second prolongation becomes

$$\begin{aligned} \mathbf{X}^3 &= \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} - \dot{\phi} \sin \phi \frac{\partial}{\partial \dot{\phi}} + \left( \dot{\theta} \sin \phi \csc^2 \theta - \dot{\phi} \cot \theta \cos \phi \right) \frac{\partial}{\partial \dot{\theta}} \\ &\quad - \left( \dot{\phi}^2 \cos \phi + \ddot{\phi} \sin \phi \right) \frac{\partial}{\partial \ddot{\phi}} + (-2\dot{\theta}^2 \sin \phi \csc^2 \theta \cot \theta + \dot{\phi}^2 \cot \theta \sin \phi \\ &\quad + 2\dot{\theta}\dot{\phi} \cos \phi \csc^2 \theta + \ddot{\theta} \csc^2 \theta \sin \phi - \ddot{\phi} \cot \theta \cos \phi) \frac{\partial}{\partial \ddot{\theta}}. \end{aligned} \quad (2.43)$$



$$(v) \quad \mathbf{X}^4 = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}$$

Here  $\xi = 0$ ,  $\eta^1 = \sin \phi$ ,  $\eta^2 = \cot \theta \cos \phi$ , therefore, equations (2.9) -(2.12) give

$$\begin{aligned} \eta_{,s}^1 &= \dot{\phi} \cos \phi, \\ \eta_{,s}^2 &= -\dot{\theta} \csc^2 \theta \cos \phi - \dot{\phi} \cot \theta \sin \phi, \\ \eta_{,ss}^1 &= -\dot{\phi}^2 \sin \phi + \ddot{\phi} \cos \phi, \\ \eta_{,ss}^2 &= \dot{\theta}^2 (2 \csc^2 \theta \cot \theta \cos \phi) + \dot{\phi}^2 (-\cot \theta \cos \phi) + 2\dot{\theta}\dot{\phi} (\csc^2 \theta \sin \phi) \\ &\quad - \ddot{\theta} \csc^2 \theta \cos \phi - \ddot{\phi} \cot \theta \sin \phi, \end{aligned}$$

so that the second prolongation becomes

$$\begin{aligned} \mathbf{X}^4 &= \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} + \dot{\phi} \cos \phi \frac{\partial}{\partial \dot{\theta}} - \left( \dot{\theta} \csc^2 \theta \cos \phi + \dot{\phi} \cot \theta \sin \phi \right) \frac{\partial}{\partial \dot{\phi}} \\ &\quad + \left( \ddot{\phi} \cos \phi - \dot{\phi}^2 \sin \phi \right) \frac{\partial}{\partial \ddot{\theta}} + (2\dot{\theta}^2 \csc^2 \theta \cot \theta \cos \phi + -\dot{\phi}^2 \cot \theta \cos \phi \\ &\quad + 2\dot{\theta}\dot{\phi} \csc^2 \theta \sin \phi - \ddot{\theta} \csc^2 \theta \cos \phi - \ddot{\phi} \cot \theta \sin \phi) \frac{\partial}{\partial \ddot{\phi}}. \end{aligned} \quad (2.44)$$

Now, we approximate the geodesic equations of a sphere by using two arbitrary functions  $f(\theta)$ ,  $g(\phi)$  and a small parameter  $\epsilon$  as

$$\begin{aligned} E_1^1 &: \ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta) = 0, \\ E_1^2 &: \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\phi) = 0, \end{aligned} \quad (2.45)$$

where

$$\begin{aligned} E_0^1 &: \ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0, \\ E_0^2 &: \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0, \end{aligned} \quad (2.46)$$

are the exact geodesic equations. For Case (i) the auxiliary functions become

$$\begin{aligned} H_1 &= \frac{1}{\epsilon} \mathbf{X}_0^0 (\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta))|_{\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta)=0} \\ &= \frac{1}{\epsilon} \left( \frac{\partial}{\partial s} \right) (\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta))|_{\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta)=0} \\ &= \frac{1}{\epsilon} (0) = 0, \\ H_2 &= \frac{1}{\epsilon} \mathbf{X}_0^0 (\ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\phi))|_{\ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\phi)=0} \\ &= \frac{1}{\epsilon} \left( \frac{\partial}{\partial s} \right) (\ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\phi))|_{\ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\phi)=0} \\ &= \frac{1}{\epsilon} (0) = 0. \end{aligned} \quad (2.47)$$

Hence  $H_1 = 0 = H_2$ , so we cannot proceed further.

Now we take Case (ii). Here the auxiliary functions become

$$\begin{aligned}
H_1 &= \frac{1}{\epsilon} \mathbf{X}_0^1 (\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta)) \big|_{\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta) = 0} \\
&= \frac{1}{\epsilon} \left( s \frac{\partial}{\partial s} - \dot{\theta} \frac{\partial}{\partial \dot{\theta}} - \dot{\phi} \frac{\partial}{\partial \dot{\phi}} - 2\ddot{\theta} \frac{\partial}{\partial \ddot{\theta}} - 2\ddot{\phi} \frac{\partial}{\partial \ddot{\phi}} \right) \\
&\quad \times \left( \ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta) \right) \big|_{\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta) = 0} \\
&= \frac{1}{\epsilon} \left( -2\ddot{\theta} + 2 \sin \theta \cos \theta \dot{\phi}^2 + \epsilon (0) \right) \big|_{\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta) = 0} \\
&= \frac{1}{\epsilon} (-2(-\epsilon f(\theta))) = 2f(\theta), \\
\\
H_2 &= \frac{1}{\epsilon} \mathbf{X}_0^1 \left( \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\phi) \right) \big|_{\ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\phi) = 0} \\
&= \frac{1}{\epsilon} \left( s \frac{\partial}{\partial s} - \dot{\theta} \frac{\partial}{\partial \dot{\theta}} - \dot{\phi} \frac{\partial}{\partial \dot{\phi}} - 2\ddot{\theta} \frac{\partial}{\partial \ddot{\theta}} - 2\ddot{\phi} \frac{\partial}{\partial \ddot{\phi}} \right) \\
&\quad \times \left( \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\phi) \right) \big|_{\ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\phi) = 0} \\
&= \frac{1}{\epsilon} \left( -2\ddot{\phi} - 2 \cot \theta \dot{\theta} \dot{\phi} - 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon (0) \right) \big|_{\ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\phi) = 0} \\
&= \frac{1}{\epsilon} (-2(-\epsilon g(\phi))) = 2g(\phi). \tag{2.48}
\end{aligned}$$

Solving for  $X_1^1$  in

$$\begin{aligned}
\mathbf{X}_1 E_0^1 \big|_{E_0^1 = 0} + H_1 &= 0, \\
\mathbf{X}_1 E_0^2 \big|_{E_0^2 = 0} + H_2 &= 0, \tag{2.49}
\end{aligned}$$

gives the same equations as (2.17)-(2.30) with a change in (2.15) and (2.16) given by

$$\eta_{ss}^1 + 2f(\theta) = 0, \tag{2.50}$$

$$\eta_{ss}^2 + 2g(\phi) = 0. \tag{2.51}$$

Equations (2.25), (2.26) and (2.29) yield

$$\xi = b_1(s). \tag{2.52}$$

Substituting the value of  $\xi$  into (2.21) and solving with (2.50), we get

$$\eta^1 = f_1(\theta, \phi)s + f_2(\theta, \phi) - f(\theta)s^2. \tag{2.53}$$

Integrating (2.53) twice with respect to  $\theta$ , and using (2.21), we get

$$\eta_{\theta\theta}^1 = f_{1\theta\theta}(\theta, \phi)s + f_{2\theta\theta}(\theta, \phi) - f_{\theta\theta}(\theta)s^2 = 0. \quad (2.54)$$

Comparing the coefficients of the powers of  $s$ , we obtain

$$f_{1\theta\theta} = 0, f_{2\theta\theta} = 0, f_{\theta\theta} = 0, \quad (2.55)$$

which implies that  $f_1$ ,  $f_2$  and  $f$  are linear in  $\theta$ . Hence we can write

$$\begin{aligned} f_1(\theta, \phi) &= f_1(\phi)\theta + f_3(\phi), \\ f_2(\theta, \phi) &= f_2(\phi)\theta + f_4(\phi), \\ f(\theta) &= c\theta + d, \end{aligned} \quad (2.56)$$

so that (2.53) becomes

$$\eta^1 = [f_1(\phi)s + f_2(\phi)]\theta + f_3(\phi)s + f_4(\phi) - (c\theta + d)s^2. \quad (2.57)$$

From (2.51), (2.18) and (2.22), we obtain

$$\eta_s^2 = 0, \quad (2.58)$$

therefore (2.51) gives

$$\eta_{ss}^2 + 2g(\phi) = 0, \quad (2.59)$$

so that

$$g(\phi) = 0, \quad (2.60)$$

$$\eta^2 = -\cot\theta f_6(\phi) + f_7(\phi). \quad (2.61)$$

Substituting the values of  $\eta^1$  and  $\eta^2$  into (2.19) we get

$$\eta_{s\phi}^1 = 0, f_1(\phi) = A, f_3(\phi) = B, \quad (2.62)$$

which gives

$$\eta^1 = [As + f_2(\phi)]\theta + Bs + f_4(\phi) - (c\theta + d)s^2. \quad (2.63)$$

Thus (2.17) yields

$$b_1(s) = As^2 - \frac{2}{3}cs^3 + c_1s + c_0, \quad (2.64)$$

so that (2.52) becomes

$$\xi = As^2 - \frac{2}{3}cs^3 + c_1s + c_0. \quad (2.65)$$

Now, inserting the values of  $\eta^1$  and  $\eta^2$  in (2.23) and (2.27) gives

$$\begin{aligned} f_2'(\phi) - \cot \theta f_6(\phi) - \cot \theta f_2'(\phi) \theta - \cot \theta f_4'(\phi) &= 0, \\ f_2''(\phi) \theta + f_4''(\phi) + \frac{1}{2} s A \sin 2\theta + \frac{1}{2} \sin 2\theta f_2(\phi) - \frac{1}{2} s^2 c \sin 2\theta \\ - s \theta A \cos 2\theta - \theta \cos 2\theta f_2(\phi) - s B \cos 2\theta - \cos 2\theta f_4(\phi) \\ + \cos 2\theta (c\theta + d) s^2 + 2 \cos^2 \theta f_6'(\phi) - \sin 2\theta f_7'(\phi) &= 0. \end{aligned}$$

Comparing the coefficients of the powers of  $\theta$  and  $s$ , we obtain

$$c = 0, d = 0, f(\theta) = 0. \quad (2.66)$$

From equations (2.60) and (2.66) it is clear that, for this case, we do not have any new symmetry. Similarly, we see that Cases (iii), (iv) and (v) also do not give any non-trivial symmetry.

## 2.2 Approximate symmetries: Second approach

Now we adopt another approach and take a more general function to make the exact geodesic equations (2.46) perturbed, and investigate the existence of approximate symmetries. Here we approximate the geodesic equations of a sphere as

$$\begin{aligned} E_1^1 : \ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta, \phi, \dot{\theta}, \dot{\phi}) &= 0, \\ E_1^2 : \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\theta, \phi, \dot{\theta}, \dot{\phi}) &= 0. \end{aligned} \quad (2.67)$$

As these are second order ordinary differential equations, we apply the second prolongation

$$\mathbf{X} = \xi \frac{\partial}{\partial s} + \eta^1 \frac{\partial}{\partial \theta} + \eta^2 \frac{\partial}{\partial \phi} + \eta_{,s}^1 \frac{\partial}{\partial \dot{\theta}} + \eta_{,s}^2 \frac{\partial}{\partial \dot{\phi}} + \eta_{,ss}^1 \frac{\partial}{\partial \ddot{\theta}} + \eta_{,ss}^2 \frac{\partial}{\partial \ddot{\phi}}, \quad (2.68)$$

where

$$\xi = \xi_0 + \epsilon \xi_1, \quad (2.69)$$

$$\eta = \eta_0 + \epsilon \eta_1, \quad (2.70)$$

and the infinitesimal generators, as before, are given by

$$\mathbf{X} = \mathbf{X}_0 + \epsilon \mathbf{X}_1. \quad (2.71)$$

Equations (2.67) are approximately invariant under the approximate group of transformations with the generator given in (2.68) if and only if

$$\begin{aligned} \mathbf{X} E^1 \big|_{E^1=0=E^2} &= 0, \\ \mathbf{X} E^2 \big|_{E^1=0=E^2} &= 0, \end{aligned} \quad (2.72)$$

so that we have

$$\begin{aligned} \left[ \mathbf{X} \left( \ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta, \phi, \dot{\theta}, \dot{\phi}) \right) \right] \big|_{\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 + \epsilon f(\theta, \phi, \dot{\theta}, \dot{\phi}) = 0} &= 0, \\ \left[ \mathbf{X} \left( \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\theta, \phi, \dot{\theta}, \dot{\phi}) \right) \right] \big|_{\ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g(\theta, \phi, \dot{\theta}, \dot{\phi}) = 0} &= 0. \end{aligned} \quad (2.73)$$

Now, let us take the following form of the functions  $f$  and  $g$  introduced in (2.67)

$$\begin{aligned} f(\theta, \phi, \dot{\theta}, \dot{\phi}) &= k_1 + k_2 \theta + k_3 \phi + k_4 \dot{\theta} + k_5 \dot{\phi} + k_6 \dot{\theta}^2 + k_7 \dot{\phi}^2, \\ g(\theta, \phi, \dot{\theta}, \dot{\phi}) &= h_1 + h_2 \theta + h_3 \phi + h_4 \dot{\theta} + h_5 \dot{\phi} + h_6 \dot{\theta}^2 + h_7 \dot{\phi}^2, \end{aligned} \quad (2.74)$$

where  $k_1, \dots, k_7, h_1, \dots, h_7$  are all constants. Then (2.73) yields

$$\begin{aligned} [\eta_{ss}^1 - 2 \sin \theta \cos \theta \dot{\phi} \eta_{s,s}^2 - \eta^1 (\cos^2 \theta - \sin^2 \theta) \dot{\phi}^2 + \epsilon(k_2 \eta^1 + k_3 \eta^2 \\ + k_4 \eta_{s,s}^1 + k_5 \eta_{s,s}^2 + 2k_6 \dot{\theta} \eta_{s,s}^1 + 2k_7 \dot{\phi} \eta_{s,s}^2)]|_{E_1=0=E_2} &= 0, \\ [\eta_{ss}^2 + 2 \cot \theta (\dot{\phi} \eta_{s,s}^1 + \dot{\theta} \eta_{s,s}^2) - 2 \dot{\theta} \dot{\phi} \csc^2 \theta \eta^1 + \epsilon(h_2 \eta^1 + h_3 \eta^2 \\ + h_4 \eta_{s,s}^1 + h_5 \eta_{s,s}^2 + 2h_6 \dot{\theta} \eta_{s,s}^1 + 2h_7 \dot{\phi} \eta_{s,s}^2)]|_{E_1=0=E_2} &= 0. \end{aligned} \quad (2.75)$$

Inserting (2.9) -(2.12) into (2.75), we obtain

$$\begin{aligned} [\eta_{ss}^1 + \dot{\theta} (2\eta_{s\theta}^1 - \xi_{ss}) + \dot{\theta}^2 (\eta_{\theta\theta}^1 - 2\xi_{s\theta}) - \dot{\theta}^3 \xi_{\theta\theta} + 2\dot{\phi} \eta_{s\phi}^1 + \dot{\phi}^2 \eta_{\phi\phi}^1 \\ + 2\dot{\theta} \dot{\phi} (\eta_{\theta\phi}^1 - \xi_{s\phi}) - 2\dot{\theta}^2 \dot{\phi} \xi_{\theta\phi} - \dot{\theta} \dot{\phi}^2 \xi_{\phi\phi} + \ddot{\theta} (\eta_{\theta}^1 - 2\xi_s - 3\dot{\theta} \xi_{\theta}) \\ - 2\ddot{\theta} \dot{\phi} \xi_{\phi} + \ddot{\phi} \eta_{\phi}^1 - \dot{\theta} \ddot{\phi} \xi_{\phi}] - \sin 2\theta \dot{\phi} [\eta_s^2 + \dot{\phi} (\eta_{\phi}^2 - \xi_s) + \dot{\theta} \eta_{\theta}^2 - \dot{\phi}^2 \xi_{\phi} \\ - \dot{\theta} \dot{\phi} \xi_{\theta}] - \eta^1 \cos 2\theta \dot{\phi}^2|_{E_1=0=E_2} &= 0, \\ [\eta_{ss}^2 + 2\dot{\theta} \eta_{s\theta}^2 + \dot{\theta}^2 \eta_{\theta\theta}^2 + \dot{\phi} (2\eta_{s\phi}^2 - \xi_{ss}) + \dot{\phi}^2 (\eta_{\phi\phi}^2 - 2\xi_{s\phi}) - \dot{\phi}^3 \xi_{\phi\phi} + 2\dot{\theta} \dot{\phi} (\eta_{\theta\phi}^2 - \xi_{s\theta}) \\ - \dot{\theta}^2 \dot{\phi} \xi_{\theta\theta} - 2\dot{\theta} \dot{\phi}^2 \xi_{\theta\phi} + \ddot{\theta} \eta_{\theta}^2 - \ddot{\theta} \dot{\phi} \xi_{\theta} + \ddot{\phi} (\eta_{\phi}^2 - 2\xi_s - 3\dot{\phi} \xi_{\phi}) - 2\dot{\theta} \ddot{\phi} \xi_{\theta}] + 2 \cot \theta \\ (\dot{\phi} \{ \eta_s^1 + \dot{\theta} (\eta_{\theta}^1 - \xi_s) - \dot{\theta}^2 \xi_{\theta} + \dot{\phi} \eta_{\phi}^1 - \dot{\theta} \dot{\phi} \xi_{\phi} \} + \dot{\theta} \{ \eta_s^2 + \dot{\phi} (\eta_{\phi}^2 - \xi_s) \\ + \dot{\theta} \eta_{\theta}^2 - \dot{\phi}^2 \xi_{\phi} - \dot{\theta} \dot{\phi} \xi_{\theta} \}) - 2\dot{\theta} \dot{\phi} \csc^2 \theta \eta^1|_{E_1=0=E_2} &= 0. \end{aligned} \quad (2.76)$$

Inserting the values of  $\ddot{\theta}$  and  $\ddot{\phi}$  from the geodesic equations (2.67), the above equations take the form

$$\begin{aligned} \eta_{ss}^1 + \dot{\theta} (2\eta_{s\theta}^1 - \xi_{ss}) + \dot{\theta}^2 (\eta_{\theta\theta}^1 - 2\xi_{s\theta}) - \dot{\theta}^3 \xi_{\theta\theta} + 2\dot{\phi} \eta_{s\phi}^1 + \dot{\phi}^2 \eta_{\phi\phi}^1 \\ + 2\dot{\theta} \dot{\phi} (\eta_{\theta\phi}^1 - \xi_{s\phi}) - 2\dot{\theta}^2 \dot{\phi} \xi_{\theta\phi} - \dot{\theta} \dot{\phi}^2 \xi_{\phi\phi} + (\sin \theta \cos \theta \dot{\phi}^2 - \epsilon f) \\ (\eta_{\theta}^1 - 2\xi_s - 3\dot{\theta} \xi_{\theta}) - 2(\sin \theta \cos \theta \dot{\phi}^2 - \epsilon f) \dot{\phi} \xi_{\phi} - (2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g) \eta_{\phi}^1 \\ + \dot{\theta} (2 \cot \theta \dot{\theta} \dot{\phi} + \epsilon g) \xi_{\phi} - \sin 2\theta \dot{\phi} [\eta_s^2 + \dot{\phi} (\eta_{\phi}^2 - \xi_s) + \dot{\theta} \eta_{\theta}^2 - \dot{\phi}^2 \xi_{\phi} - \dot{\theta} \dot{\phi} \xi_{\theta} \\ - \eta^1 \cos 2\theta \dot{\phi}^2 + \epsilon(k_2 \eta^1 + k_3 \eta^2 + (k_4 + 2k_6 \dot{\theta})(\eta_s^1 + \dot{\theta} (\eta_{\theta}^1 - \xi_s) - \dot{\theta}^2 \xi_{\theta} + \dot{\phi} \eta_{\phi}^1 \\ - \dot{\theta} \dot{\phi} \xi_{\phi}) + (k_5 + 2k_7 \dot{\phi})(\eta_s^2 + \dot{\phi} (\eta_{\phi}^2 - \xi_s) + \dot{\theta} \eta_{\theta}^2 - \dot{\phi}^2 \xi_{\phi} - \dot{\theta} \dot{\phi} \xi_{\theta})) = 0, \end{aligned}$$

$$\begin{aligned}
& \eta_{ss}^2 + 2\dot{\theta}\eta_{s\theta}^2 + \dot{\theta}^2\eta_{\theta\theta}^2 + \dot{\phi}(2\eta_{s\phi}^2 - \xi_{ss}) + \dot{\phi}^2(\eta_{\phi\phi}^2 - 2\xi_{s\phi}) - \dot{\phi}^3\xi_{\phi\phi} + 2\dot{\theta}\dot{\phi} \\
& (\eta_{\theta\phi}^2 - \xi_{s\theta}) - \dot{\theta}^2\dot{\phi}\xi_{\theta\theta} - 2\dot{\theta}\dot{\phi}^2\xi_{\theta\phi} + (\sin\theta\cos\theta\dot{\phi}^2 - \epsilon f)\eta_{\theta}^2 - (\sin\theta\cos\theta\dot{\phi}^2 \\
& - \epsilon f)\dot{\phi}\xi_{\theta} - (2\cot\theta\dot{\theta}\dot{\phi} + \epsilon g)(\eta_{\phi}^2 - 2\xi_s - 3\dot{\phi}\xi_{\phi}) + 2\dot{\theta}(2\cot\theta\dot{\theta}\dot{\phi} + \epsilon g)\xi_{\theta} + 2\cot\theta \\
& (\dot{\phi}\{\eta_s^1 + \dot{\theta}(\eta_{\theta}^1 - \xi_s) - \dot{\theta}^2\xi_{\theta} + \dot{\phi}\eta_{\phi}^1 - \dot{\theta}\dot{\phi}\xi_{\phi}\} + \dot{\theta}\{\eta_s^2 + \dot{\phi}(\eta_{\phi}^2 - \xi_s) + \dot{\theta}\eta_{\theta}^2 \\
& - \dot{\phi}^2\xi_{\phi} - \dot{\theta}\dot{\phi}\xi_{\theta}\}) - 2\dot{\theta}\dot{\phi}\csc^2\theta\eta^1 + \epsilon(h_2\eta^1 + h_3\eta^2 + (h_4 + 2h_6\dot{\theta})(\eta_s^1 + \dot{\theta}(\eta_{\theta}^1 - \xi_s) \\
& - \dot{\theta}^2\xi_{\theta} + \dot{\phi}\eta_{\phi}^1 - \dot{\theta}\dot{\phi}\xi_{\phi}) + (h_5 + 2h_7\dot{\phi})(\eta_s^2 + \dot{\phi}(\eta_{\phi}^2 - \xi_s) + \dot{\theta}\eta_{\theta}^2 - \dot{\phi}^2\xi_{\phi} - \dot{\theta}\dot{\phi}\xi_{\theta})) = \textcolor{black}{(2.77)}
\end{aligned}$$

Now, we use (2.69) and (2.70) in (2.77), and separate terms with  $\epsilon^0$  and  $\epsilon^1$ , and ignoring terms with  $\epsilon^2$ . For  $\epsilon^0$  we get

$$\begin{aligned}
& \eta_{0ss}^1 + \dot{\theta}(2\eta_{s\theta}^1 - \xi_{0ss}) + \dot{\theta}^2(\eta_{0\theta\theta}^1 - 2\xi_{s\theta}) - \dot{\theta}^3\xi_{0\theta\theta} + 2\dot{\phi}\eta_{0s\phi}^1 + \dot{\phi}^2\eta_{0\phi\phi}^1 \\
& + 2\dot{\theta}\dot{\phi}(\eta_{0\theta\phi}^1 - \xi_{0s\phi}) - 2\dot{\theta}^2\dot{\phi}\xi_{0\theta\phi} - \dot{\theta}\dot{\phi}^2\xi_{0\phi\phi} + \sin\theta\cos\theta\dot{\phi}^2(\eta_{0\theta}^1 \\
& - 2\xi_s - 3\dot{\theta}\xi_{0\theta}) - 2\sin\theta\cos\theta\dot{\phi}^3\xi_{0\phi} - 2\cot\theta\dot{\theta}\dot{\phi}\eta_{0\phi}^1 + 2\cot\theta\dot{\theta}^2\dot{\phi}\xi_{0\phi} \\
& - \sin 2\theta\dot{\phi}[\eta_{0s}^2 + \dot{\phi}(\eta_{0\phi}^2 - \xi_{0s}) + \dot{\theta}\eta_{0\theta}^2 - \dot{\phi}^2\xi_{0\phi} - \dot{\theta}\dot{\phi}\xi_{0\theta}] - \eta_0^1\cos 2\theta\dot{\phi}^2 = 0, \\
& \eta_{0ss}^2 + 2\dot{\theta}\eta_{0s\theta}^2 + \dot{\theta}^2\eta_{0\theta\theta}^2 + \dot{\phi}(2\eta_{s\phi}^2 - \xi_{0ss}) + \dot{\phi}^2(\eta_{0\phi\phi}^2 - 2\xi_{s\phi}) - \dot{\phi}^3\xi_{0\phi\phi} \\
& + 2\dot{\theta}\dot{\phi}(\eta_{0\theta\phi}^2 - \xi_{0s\theta}) - \dot{\theta}^2\dot{\phi}\xi_{0\theta\theta} - 2\dot{\theta}\dot{\phi}^2\xi_{0\theta\phi} + \sin\theta\cos\theta\dot{\phi}^2\eta_{0\theta}^2 - \sin\theta\cos\theta\dot{\phi}^3\xi_{0\theta} \\
& - 2\cot\theta\dot{\theta}\dot{\phi}(\eta_{0\phi}^2 - 2\xi_s - 3\dot{\phi}\xi_{0\phi}) + 4\cot\theta\dot{\theta}^2\dot{\phi}\xi_{0\phi} \\
& + 2\cot\theta(\dot{\phi}\{\eta_{0s}^1 + \dot{\theta}(\eta_{0\theta}^1 - \xi_{0s}) - \dot{\theta}^2\xi_{0\theta} + \dot{\phi}\eta_{0\phi}^1 - \dot{\theta}\dot{\phi}\xi_{0\phi}\} \\
& + \dot{\theta}\{\eta_{0s}^2 + \dot{\phi}(\eta_{0\phi}^2 - \xi_{0s}) + \dot{\theta}\eta_{0\theta}^2 - \dot{\phi}^2\xi_{0\phi} - \dot{\theta}\dot{\phi}\xi_{0\theta}\}) - 2\dot{\theta}\dot{\phi}\csc^2\theta\eta^1 = 0. \textcolor{black}{(2.78)}
\end{aligned}$$

Comparing the coefficients of the powers of  $\dot{\theta}$  and  $\dot{\phi}$ , we obtain a system of partial differential equations, which on solving yields the following solution

$$\xi_0 = c_1 s + c_0, \quad (2.79)$$

$$\eta_0^1 = c_3 \cos \phi + c_4 \sin \phi, \quad (2.80)$$

$$\eta_0^2 = \cot \theta (c_4 \cos \phi - c_3 \sin \phi) + c_2. \quad (2.81)$$

We follow the same procedure for  $\epsilon^1$ , compare the coefficients of the powers of  $\dot{\theta}$  and  $\dot{\phi}$ , and obtain a system of partial differential equations. We denote

$$\xi_{0s} = A, \eta_0^1 = B, \eta_0^2 = C, \eta_{0\phi}^1 = D, \eta_{0\phi}^2 = E, \eta_{0\theta}^2 = F, \quad (2.82)$$

so that the system of equations become

$$(\dot{\theta}\dot{\phi})^0 : \eta_{1ss}^1 + 2(k_1 + k_2\theta + k_3\phi)A - (h_1 + h_2\theta + h_3\phi)D + k_2B + k_3C = 0, \quad (2.83)$$

$$\eta_{ss}^2 - (k_1 + k_2\theta + k_3\phi)F - (h_1 + h_2\theta + h_3\phi)(E - 2A) + h_2B + h_3C = 0, \quad (2.84)$$

$$\dot{\theta} : 2\eta_{1s\theta}^1 - \xi_{1ss} + k_4A - h_4D + k_5F = 0, \quad (2.85)$$

$$2\eta_{1s\theta}^2 + 2\cot\theta\eta_{1s}^2 - k_4F - h_4E + h_4A + h_5F = 0, \quad (2.86)$$

$$\dot{\phi} : 2\eta_{1s\phi}^1 - \sin 2\theta\eta_{1s}^2 + k_5A - h_5D + k_4D + k_5E = 0, \quad (2.87)$$

$$2\eta_{1s\phi}^2 - \xi_{1ss} + 2\cot\theta\eta_{1s}^1 - k_5F + h_5A + h_4D = 0, \quad (2.88)$$

$$\dot{\theta}^2 : \eta_{1\theta\theta}^1 - 2\xi_{1s\theta} - h_6D = 0, \quad (2.89)$$

$$\eta_{1\theta\theta}^2 + 2\cot\theta\eta_{1\theta}^2 - k_6F - h_6E = 0, \quad (2.90)$$

$$\dot{\phi}^2 : \eta_{1\phi\phi}^1 + \sin\theta\cos\theta\eta_{1\theta}^1 - \eta_1^1\cos 2\theta - \sin 2\theta\eta_{1\phi}^2 - h_7D + 2k_7E = 0, \quad (2.91)$$

$$\eta_{1\phi\phi}^2 - 2\xi_{1s\phi} + \sin\theta\cos\theta\eta_{1\theta}^2 + 2\cot\theta\eta_{1\phi}^1 - k_7F + h_7E = 0, \quad (2.92)$$

$$\dot{\theta}^3 : \xi_{1\theta\theta} = 0, \quad (2.93)$$

$$\dot{\phi}^3 : -\xi_{1\phi\phi} - \sin\theta\cos\theta\xi_{1\theta} = 0, \quad (2.94)$$

$$\dot{\theta}\dot{\phi} : 2(\eta_{1\theta\phi}^1 - \xi_{1s\phi}) - \sin 2\theta\eta_{1\theta}^2 - 2\cot\theta\eta_{1\phi}^1 + 2k_6D + 2k_7F = 0, \quad (2.95)$$

$$2(\eta_{1\theta\phi}^2 - \xi_{1s\theta}) + 2\cot\theta\eta_{1\theta}^1 - 2\csc^2\theta\eta_1^1 + 2h_6D + 2h_7F = 0, \quad (2.96)$$

$$\dot{\theta}\dot{\phi}^2 : -\xi_{1\phi\phi} - \sin\theta\cos\theta\xi_{1\theta} = 0, \quad (2.97)$$

$$-2\xi_{1\theta\phi} + 2\cot\theta\xi_{1\phi} = 0. \quad (2.98)$$

Equations (2.93), (2.94) and (2.97) yield

$$\xi_1 = b_1(s). \quad (2.99)$$

Substituting the value of  $\xi$  into (2.89) and solving with (2.83), we get

$$k_3 = 0, \quad (2.100)$$

$$\begin{aligned}\eta_1^1 = & -(k_1 + k_2\theta)AS^2 + \frac{1}{2}(h_1 + h_2\theta + h_3\phi)Ds^2 \\ & - \frac{1}{2}k_2Bs^2 + (g_1(\phi)\theta + g_2(\phi))s + g_3(\phi)\theta + g_4(\phi) + \frac{1}{2}h_6D^2,\end{aligned}\quad (2.101)$$

where  $g_1(\phi)$ ,  $g_2(\phi)$ ,  $g_3(\phi)$  and  $g_4(\phi)$  are functions of integration. From (2.84) and (2.86), we obtain

$$k_1 = k_2 = k_3 = h_1 = h_2 = h_3 = 0, \quad (2.102)$$

$$\eta_1^2 = \frac{1}{2}s[-\cot\theta(k_4F + h_4E - h_4A - h_5F) - k_4F_\theta - h_4E_\theta + h_5F_\theta] + f_3(\theta, \phi), \quad (2.103)$$

where  $f_3(\theta, \phi)$  is a function of integration. Using the above value of  $\eta_1^2$  in (2.90), we get

$$k_4 = h_4 = h_5 = 0, \quad (2.104)$$

$$\eta_1^2 = (\theta \cot\theta - \ln\sin\theta)k_6D - \frac{1}{2}\theta h_6D - \cot\theta g_5(\phi) + g_6(\phi), \quad (2.105)$$

where  $g_5(\phi)$  and  $g_6(\phi)$  are functions of integration. Using the values of  $\eta_1^1$  and  $\eta_1^2$  in (2.87), gives

$$\begin{aligned}g_1(\phi) &= c_5, \\ g_2(\phi) &= -\frac{1}{2}k_5(C + A\phi) + c_6.\end{aligned}$$

Now (2.88) implies

$$k_5 = 0, \quad (2.106)$$

$$c_5 = 0, \quad (2.107)$$

$$\xi = c_7s + c_8. \quad (2.108)$$

Similarly (2.95) and (2.91) give

$$h_6 = 0, \quad (2.109)$$

$$g_3(\phi) = k_6B + c_9, \quad (2.110)$$

$$k_6 = 0, \quad (2.111)$$

$$c_9 = 0, \quad (2.112)$$

$$g_4(\phi) = c_{10}\cos\phi + c_{11}\sin\phi - \frac{\phi}{2}(k_7\tan\theta F + h_7B), \quad (2.113)$$

$$\begin{aligned}g_5(\phi) &= c_{10}\sin\phi - c_{11}\cos\phi - \frac{k_7}{2}\tan\theta\csc^2\theta D \\ &\quad - \frac{h_7}{2}B - \frac{h_7}{2}\phi D + k_7\tan\theta F\left(\frac{\phi}{2} + 1\right),\end{aligned}\quad (2.114)$$



$$g_{4\phi\phi} - \cos 2\theta g_4 + 2 \cos^2 \theta g_{5\phi} - \sin 2\theta g_{6\phi} - h_7 D + 2k_7 E = 0. \quad (2.115)$$

Again, using the values of  $\eta_1^1$  and  $\eta_1^2$  in (2.96) and (2.92), we get

$$g_6(\phi) = c_{12}, \quad (2.116)$$

$$h_7 = 0, \quad (2.117)$$

$$k_7 = 0. \quad (2.118)$$

Therefore, we have

$$\xi_1 = c_7 s + c_8, \quad (2.119)$$

$$\eta_1^1 = c_{10} \cos \phi + c_{11} \sin \phi, \quad (2.120)$$

$$\eta_1^2 = \cot \theta (c_{11} \cos \phi - c_{10} \sin \phi) + c, \quad (2.121)$$

which are trivial, as all the constants  $k_1, \dots, k_7, h_1, \dots, h_7$  in (2.74) become zero. Hence we see that this case also does not give any non-trivial approximate symmetries.

### 3. Conclusion

One of the methods to solve differential equations is the symmetry method by which we can solve differential equations, reduce the order of an ordinary differential equation and can reduce the number of independent variables in a partial differential equation if it is invariant under a one-parameter Lie group of point transformations. The method of ‘canonical variables’ is one of the basic procedures for the integration of ordinary differential equations with known symmetries. Conservation laws for the Euler-Lagrange equations can be constructed when their symmetries are known and hence Lagrangians can be constructed.

For tackling differential equations with a small parameter, the method of ‘approximate symmetries’ can be used. Considering the geodesic equations of a sphere as a system of two ordinary differential equations we have investigated their approximate symmetries. Two different approaches have been adopted to find these approximate symmetries. Firstly, by converting the geodesic equations into perturbed equations by adding the general functions  $f(\theta)$  and  $g(\phi)$  and taking a symmetry of the geodesic equations of the sphere. In the second approach we use the functions

$$\begin{aligned} f(\theta, \phi, \dot{\theta}, \dot{\phi}) &= k_1 + k_2 \theta + k_3 \phi + k_4 \dot{\theta} + k_5 \dot{\phi} + k_6 \dot{\theta}^2 + k_7 \dot{\phi}^2, \\ g(\theta, \phi, \dot{\theta}, \dot{\phi}) &= h_1 + h_2 \theta + h_3 \phi + h_4 \dot{\theta} + h_5 \dot{\phi} + h_6 \dot{\theta}^2 + h_7 \dot{\phi}^2, \end{aligned}$$

to convert the geodesic equations into perturbed equations. In both the cases, no non-trivial approximate symmetry has been found.

It would be interesting to see if one can obtain non-trivial approximate symmetries for these manifolds by some other method.

## Acknowledgments

The authors are thankful to Fazal M. Mahomed for useful discussions. A research grant from the Higher Education Commission of Pakistan is gratefully acknowledged.

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